

Graviton emission from the brane in the bulk in extra dimension

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Abstract

In a model of 3-brane embedded in 5D space-time we calculate the graviton emission from the brane to the bulk. Matter is confined to the brane, gravitons produced in reactions of matter on the brane escape to the bulk. The Einstein equations which are modified by the terms due to graviton production are solved perturbatively, the leading order being that without the graviton production. In the period of late cosmology, in which in the generalized Friedmann equation the term linear in the energy density of matter is dominant, we calculate the spectrum of gravitons (of the tower of Kaluza-Klein states) and the collision integral in the Boltzmann equation. We find the energy-momentum tensor of the emitted gravitons in the bulk, and using it show that corrections due to graviton production to the leading-order terms in the Einstein equations are small, and the perturbative approach is justified. We calculate the difference of abundances of ${}^4\text{He}$ produced in primordial nucleosynthesis in the models with and without the graviton production, and find that the difference is a very small number, much smaller than that estimated previously.

1 Introduction

Brane-world scenarios with a 3-brane identified with the observable Universe which is embedded in a higher-dimensional space-time provide an alternative to the standard 4D cosmology [1, 2, 3, 4, 5], reviews [6, 7, 8]. A necessary requirement on these models is that they should reproduce the main observational cosmological data, the age of the Universe, abundances of elements produced in primordial nucleosynthesis, etc. A general property of the models with extra dimensions is that gravity propagates in the extra dimensions independent of whether the ordinary matter is confined to the brane or not. This entails a peculiar property of the models with extra dimensions which is absent in the standard cosmology: gravitons which are produced in reactions of particles of matter on the brane can escape from the brane and propagate in the bulk [9, 10, 11, 12, 14, 15]. As a consequence the Einstein equations contain terms accounting for the graviton emission. Cosmological evolution of matter on the brane is also affected by this process.

Roughly the energy loss on the brane due to the process $a + b \rightarrow G + X$ can be estimated as

$$\frac{d\hat{\rho}}{dt} = - \langle n_a n_b \sigma_{a+b \rightarrow G+X} v E_G \rangle,$$

where in the radiation-dominated period of the evolution of the Universe $n_a, n_b \sim T^3$ and $E_G \sim T$. This yields

$$\frac{d\hat{\rho}}{dt} \sim -\kappa^2 T^8.$$

Here T is temperature of the Universe and $\kappa^2 = 8\pi/M^3$, where M is the 5D Planck mass, is the 5D gravitational constant. Although at the late stages of the cosmological evolution the energy loss to the bulk is small, at high temperatures this effect can be significant.

In the present paper we consider the problem of graviton production and emission to the bulk in a model with one 3-brane embedded in the bulk with one extra dimension. We perform our calculations in a picture in which the metric is time-dependent and the brane is located at a fixed position in the extra dimension.

The Einstein equations including the terms due to graviton production are solved in the perturbative approach. In the leading order we neglect the graviton production, and include it in the next order. The classical background metric is the leading-order solution of the 5D Einstein equations. Background metric is a warped extension of the metric on the brane to the bulk with the warping factor $e^{\pm\mu y}$ [2]. Here $\mu \sim \sqrt{-\Lambda}$, where Λ/κ^2 is the 5D cosmological constant and y a coordinate of the extra dimension.

We perform calculations in the period of late cosmology. In this period in the generalized Friedmann equation

$$H^2(t) = \frac{\mu\kappa^2\hat{\rho}}{3} + \left(\frac{\kappa^2\hat{\rho}}{6}\right)^2 + \dots$$

the term linear in energy density on the brane is much larger than the quadratic term, or, equivalently, $\mu \gg \kappa^2\hat{\rho}$. For the value of μ of order $10^{-12}GeV$, which we adopt for numerical estimates, the range of temperatures of the Universe characteristic to the period of late cosmology extends to $T \sim 5 \cdot 10^2 GeV$.

Evolution of the energy density on the brane $\hat{\rho}$ is determined by the Boltzmann equation. The collision term in the Boltzmann equation accounts for the graviton emission from the brane. To calculate the collision term, solving the equations of motion for fluctuations over the background metric, we find the spectrum and the eigenfunctions of of the tower of Kaluza-Klein gravitons and the energy-momentum tensor of the emitted gravitons.

Solution of the Einstein equations requires integration of the 5D energy-momentum tensor over coordinates of the extra dimension. This, in turn, requires determination of the form of the 5D energy-momentum tensor not only on the brane, where gravitons are emitted, but also in the bulk. Integrating the geodesic equations for the null geodesics along which gravitons propagate and using conservation equations for the energy-momentum tensor in the bulk, we find the form of the energy-momentum tensor in the bulk.

Using the obtained energy-momentum tensor in the bulk, we show that in the period of late cosmology in the Friedmann equation corrections to the leading-order terms due to graviton emission are small, and the perturbative approach is consistent.

Graviton emission changes cosmological evolution of matter on the brane. Time (temperature) dependence of of the Hubble function determined from the Friedmann equation which includes the terms due to graviton emission is different from that in the standard cosmological model. This, in turn, results in a change of abundances of light elements produced in primordial nucleosynthesis [16]. We find that the difference of abundances of 4He produced in primordial nucleosynthesis calculated in the models with and without the graviton production is a small number, much smaller than that estimated in [10, 11, 12]. A crude estimate of the graviton production in the early cosmological period, which takes into account the bounce of produced gravitons back to the brane [12], does not alter this result.

2 System of Einstein equations

We consider the 5D models with one 3D brane embedded in the bulk with the action

$$S_5 = \frac{1}{2\kappa^2} \left[\int_{\Sigma} d^5x \sqrt{-g^{(5)}} (R^{(5)} - 2\Lambda) + 2 \int_{\partial\Sigma} K \right] - \int_{\partial\Sigma} d^4x \sqrt{-g^{(4)}} \hat{\sigma} - \int_{\partial\Sigma} d^4x \sqrt{-g^{(4)}} L_m, \quad (1)$$

where $x_4 \equiv y$ is coordinate of the infinite extra dimension, $\kappa^2 = 8\pi/M^3$. We consider a class of metrics of the form ¹

$$ds_5^2 = g_{ij}^{(5)} dx^i dx^j = -n^2(y, t) dt^2 + a^2(y, t) \eta_{ab} dx^a dx^b + dy^2 \equiv dy^2 + g_{\mu\nu} dx^\mu dx^\nu. \quad (2)$$

The brane is spatially flat and is located at the fixed position $y = 0$. The 5D Einstein equations are

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R = \kappa^2 T_{ij}^{(5)} - g_{ij} \Lambda - \delta_{ij}^{\mu\nu} \frac{\sqrt{-g^{(4)}}}{\sqrt{-g^{(5)}}} \delta(y) g_{\mu\nu} \kappa^2 \hat{\sigma}. \quad (3)$$

Here $T_{ij}^{(5)}$ is the sum of the energy-momentum tensor of matter confined to the brane and the bulk energy-momentum tensor. The energy-momentum tensor of matter on the brane is taken in the form

$$\delta(y) \tau_\mu^\nu = \text{diag} \delta(y) \{-\hat{\rho}, \hat{p}, \hat{p}, \hat{p}\}. \quad (4)$$

The components of the Einstein tensor G_{ij}

$$G_{00} = 3 \left[\frac{\dot{a}^2}{a^2} - n^2 \left(\frac{a''}{a} + \frac{a'^2}{a^2} \right) \right] \quad (5)$$

$$G_{44} = 3 \left[\left(\frac{a'^2}{a^2} + \frac{a' n'}{a n} \right) - \frac{1}{n^2} \left(\frac{\dot{a}^2}{a^2} - \frac{\dot{a} \dot{n}}{a n} + \frac{\ddot{a}}{a} \right) \right] \quad (6)$$

$$G_{04} = 3 \left(\frac{n'}{n} \frac{\dot{a}}{a} - \frac{\dot{a}'}{a} \right) \quad (7)$$

satisfy the relations ² (cf. [2])

$$G_0^0 - G_4^0 \frac{\dot{a}}{a'} = \frac{3}{2a'a^3} F' \quad (8)$$

$$G_{44} - G_{04} \frac{a'}{\dot{a}} = \frac{3}{2\dot{a}a^3} \dot{F}, \quad (9)$$

where

$$F = (a'a)^2 - \frac{(\dot{a}a)^2}{n^2} \quad (10)$$

In the leading order we neglect emission of the gravitons produced in collisions of particles of matter on the brane in the bulk. In the next order we take into account the graviton emission into the bulk, the energy-momentum tensor in the bulk is \tilde{T}_j^i .

¹The indices i, j run over 0, ..., 4, the Greek indices are 0, ..., 3, and $a, b = 1, 2, 3$

²Prime and dot denote differentiation over y and t .

For the following it is convenient to introduce the normalized expressions for energy density, pressure and cosmological constant on the brane which all have the same dimensionality $[GeV]$

$$\mu = \sqrt{-\frac{\Lambda}{6}}, \quad \sigma = \frac{\kappa^2 \hat{\sigma}}{6}, \quad \rho = \frac{\kappa^2 \hat{\rho}}{6}, \quad p = \frac{\kappa^2 \hat{p}}{6}. \quad (11)$$

The functions $a(y, t)$ and $n(y, t)$ satisfy the junction conditions on the brane [2] ³

$$\begin{aligned} \frac{a'(0, t)}{a(0, t)} &= -\sigma - \rho(t) \\ \frac{n'(0, t)}{n(0, t)} &= 2\rho(t) + 3p(t) - \sigma \end{aligned} \quad (12)$$

Using reparametrization of t , we set $n(0, t) = 1$, i.e. t is the proper time on the brane. Eq. (9) can be rewritten as

$$\dot{F} = -\mu^2(\dot{a}^4) - \frac{\kappa^2}{6}(a^4)' \check{T}_{04} + \frac{\kappa^2}{6}(\dot{a}^4) \check{T}_{44}. \quad (13)$$

On the brane, using junction conditions and setting $\sigma \simeq \mu$, we have ⁴

$$\dot{F} = \mu^2(\dot{a}^4) + \frac{2\kappa^2 a^4}{3}(\rho + \mu) \check{T}_{04} + \frac{2\kappa^2 a^3 \dot{a}}{3} \check{T}_{44}. \quad (14)$$

Integrating (14) in the interval (t, t_l) , where the initial time t_l (of the onset of the period of the late cosmology) is defined below, we obtain

$$F(0, t) = \mu^2 a^4(0, t) + \frac{2\kappa^2}{3} \int_{t_l}^t dt' \check{T}_{04}(t')(\rho(t') + \mu) a^4(0, t') + \frac{2\kappa^2}{3} \int_{t_l}^t dt' \check{T}_{44}(t') \dot{a}(0, t') a^3(0, t') - C. \quad (15)$$

Substituting expression (10) for F and using the junction conditions, we rewrite (15) in a form of the generalized Friedmann equation (cf. [1, 2])

$$H^2(t) = \rho^2(t) + 2\mu\rho(t) + \mu\rho_w(t) - \frac{2\kappa^2}{3a^4(0, t)} \int_{t_l}^t dt' [\check{T}_{04}(t')(\rho(t') + \mu) + \check{T}_{44}(t')H(t')] a^4(0, t'). \quad (16)$$

Here we introduced

$$H(t) = \frac{\dot{a}(0, t)}{a(0, t)}, \quad \rho_w(t) = \rho_{w0} \left(\frac{a(0, t_0)}{a(0, t)} \right)^4 \quad (17)$$

where $\rho_w(t)$ is the so-called Weyl radiation term [7], $\mu\rho_{w0} = C/a^4(0, t_0)$. To comply with the observational data, the Weyl radiation term, which appears in the leading-order solution to the Einstein equations should be small ⁵. Assuming that the terms with the bulk energy-momentum

³We assume invariance $y \leftrightarrow -y$.

⁴From the 5D Einstein equations and the fit of the data on abundance of 4He produced in nucleosynthesis it follows that $\sigma^2 = \mu^2(1 + O(H_0^2/\mu^2))$, where H_0 is the present-time Hubble parameter [18, 19]. For $\mu \sim 10^{-12} GeV$ correction is $\sim 10^{-60}$.

⁵The Weyl radiation term ρ_w is small as compared to the radiation energy density ρ_r . BBN constraints to the ratio of the Weyl radiation term to the photon energy density at the temperatures $T \lesssim 0.8 MeV$ are $-0.4 < \rho_w/\rho_\gamma < 0.1$ [22], or $|\rho_w/\rho_\gamma| < 0.55$ [19].

tensor can be treated perturbatively, we neglect corrections to the Weyl radiation term due to graviton emission from the brane.

Substituting from the junction conditions the expressions for n'/n and a'/a at $y = 0$, we transform the (04) component of the Einstein equations (7) to the form

$$\dot{\rho} + 3H(\rho + p) = \frac{\kappa^2 \tilde{T}_{04}}{3}. \quad (18)$$

On the other hand, the same equation, which is a generalization of the conservation equation for the energy-momentum tensor of the matter confined to the brane to the case with the energy-momentum flow in the bulk, is obtained by integration across the brane of the 5D conservation law $\nabla_i T_0^i = 0$ [9].

Some relevant equations following from general approach to the 3-brane embedded in the AdS bulk are found in the appendix A.

3 Period of late cosmology

Graviton production by hot matter is sufficiently intensive in the radiation-dominated period of cosmology. In the period of late cosmology the terms linear in energy density are dominant in the Friedmann equation. Equivalently, in the radiation-dominated period of late cosmology, this can be stated as $\rho_r(t)/\mu < 1$, or

$$\frac{\rho_r(T)}{\mu} = \frac{\kappa^2 \hat{\rho}_r(T)}{6\mu} = \frac{4\pi^3 g_*(T) T^4}{90\mu M^3} \simeq \frac{4\pi^3 g_*(T) T^4}{90(\mu M_{pl})^2} < 1, \quad (19)$$

where we used that $\mu M_{pl}^2/M^3 \simeq 1$ [18, 19]. Taking $\mu \sim 10^{-12} GeV$, we find that the approximation of late cosmology is valid up to the temperatures of order $5 \cdot 10^2 GeV$.

In the leading approximation, in which the bulk energy-momentum tensor is set to zero, from the system of the 5D Einstein equations and junction conditions on the brane is obtained the unique extension of the metric the components from the brane to the bulk [2]

$$a^2(y, t) = \frac{a^2(0, t)}{4} \left[e^{2\mu|y|} \left(\left(\frac{\rho}{\mu} \right)^2 + \frac{\rho_w}{\mu} \right) + e^{-2\mu|y|} \left(\left(\frac{\rho}{\mu} + 2 \right)^2 + \frac{\rho_w}{\mu} \right) - 2 \left(\frac{\rho}{\mu} \left(\frac{\rho}{\mu} + 2 \right) + \frac{\rho_w}{\mu} \right) \right], \quad (20)$$

and

$$n(y, t) = \frac{\dot{a}(y, t)}{\dot{a}(0, t)}. \quad (21)$$

Without the Weyl radiation term the function $a^2(y, t)$ is ⁶

$$a^2(y, t) = \frac{a^2(0, t)}{4} \left[e^{2\mu|y|} \left(\frac{\rho}{\mu} \right)^2 + e^{-2\mu|y|} \left(\frac{\rho}{\mu} + 2 \right)^2 - 2 \frac{\rho}{\mu} \left(\frac{\rho}{\mu} + 2 \right) \right], \quad (22)$$

where $a(0, t)$ is a solution of the Friedmann equation on the brane. $a^2(y, t)$ (22) As a function of y $a^2(y, t)$ (22) has the minimum equal to zero at the point $|\bar{y}|$

$$e^{2\mu|\bar{y}|} = 1 + \frac{2\mu}{\rho}. \quad (23)$$

⁶The expression for $a^2(y, t)$ in [2] contained no brane tension σ .

The function $n(y, t)$ has no zeroes. At the point \bar{y} the scalar curvature $R^{(5)}$ is finite. The function (20) has no zeroes, but the function $n(y, t)$ has a zero. Again this zero is a coordinate singularity.

In the region $0 < |y| < |\bar{y}|$ and for $\rho/\mu \ll 1$ the functions $a(y, t)$ and $n(y, t)$ can be approximated as

$$\begin{aligned} a(y, t) &\simeq a(0, t)e^{-\mu|y|} \\ n(y, t) &\simeq e^{-\mu|y|}. \end{aligned} \quad (24)$$

With (24) we obtain the approximate form of the metric in the region $0 < |y| < |\bar{y}|$

$$ds^2 \simeq dy^2 + e^{-2\mu|y|}(-dt^2 + a^2(0, t)\eta_{ab}dx^a dx^b). \quad (25)$$

With the approximate metric we have $a'(0, t)/a(0, t) = -\mu$, which for $\rho/\mu \ll 1$ is close to the exact junction condition $a'(0, t)/a(0, t) = -(\mu + \rho)$.

In the radiation-dominated period solution of the Friedmann equation

$$H^2 = 2\mu\rho + \rho^2 \quad (26)$$

for the radiation energy density is

$$\rho(t) = \frac{1}{8\mu t^2 + 4t}. \quad (27)$$

In the period of late cosmology the leading term is

$$\rho(t) \simeq \frac{1}{8\mu t^2}.$$

If $\rho(t) \gg \mu$, we have $\rho(t) \simeq 1/4t$.

4 Fluctuations of background metric

To calculate the spectrum of the Kaluza-Klein tower of gravitons, we solve the field equations for fluctuations h_{ij} over the background metric g_{ij} . The purpose of this section is to clarify the issue of the gauge conditions in solving the equations for fluctuations. We show that the traceless transverse conditions on h_{ij} cannot be imposed as gauge conditions at the level of the action, but can be imposed on solutions of the equations of motion.

The part of the action quadratic in fluctuations is

$$\begin{aligned} I = \frac{1}{2} \int \sqrt{-g^{(5)}} \left[(R - \Lambda_{(5)}) \left(-\frac{1}{2} h_i^j h_j^i + \frac{1}{4} h^2 \right) - R_i^j h_j^i h + 2R_i^j h_j^k h_k^i \right. \\ \left. + \frac{1}{2} (2h_{qi;k} h^{ik;q} - h_{ik;q} h^{ik;q} + h_{,q} h^{,q} - 2h_{,i} h^{ik}_{;k}) \right]. \end{aligned} \quad (28)$$

The action I is invariant under the gauge transformations

$$\tilde{h}_{kl} = h_{kl} - (\nabla_k \xi_l + \nabla_l \xi_k), \quad (29)$$

where ∇ is defined with respect to the metric g_{ij} . With the background metric (2), $ds^2 = dy^2 + g_{\mu\nu} dx^\mu dx^\nu$, the gauge transformations (29) take the form

$$\tilde{h}_{44} = h_{44} - 2\partial_4 \xi_4 \quad (30)$$

$$\tilde{h}_{4\mu} = h_{4\mu} - (\partial_\mu \xi_4 + \partial_4 \xi_\mu - 2\Gamma_{\mu 4}^\nu \xi_\nu) \quad (31)$$

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - (D_\mu \xi_\nu + D_\nu \xi_\mu - 2\Gamma_{\mu\nu}^4 \xi_4), \quad (32)$$

where D_μ is defined with respect to the metric $g_{\mu\nu}$.

To solve the equations following from the action (28) in the region $0 < y < \bar{y}$, instead of the exact metric, we use the approximate metric (25). In the region $0 < y < \bar{y}$ we rewrite it as

$$ds^2 = dy^2 + b(y)\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu, \quad (33)$$

where $b(y) = e^{-2\mu|y|}$. We impose the gauge conditions $h_{4i} = 0$. The gauge condition $\tilde{h}_{44} = 0$ can be realized by performing transformation with

$$\xi_4(y, x^\mu) = \frac{1}{2} \int_{-y}^y dy' h_{44}(y', x^\mu) \quad (34)$$

Condition $\tilde{h}_{4\mu} = 0$ is imposed by taking

$$\xi_\mu(y, x^\rho) = b(y) \left(C_\mu(x^\rho) + \int_{-y}^y dy' b^{-1}(y') (h_{4\mu} - \partial_\mu \xi_4)(y', x^\rho) \right). \quad (35)$$

$C_\mu(x)$ are arbitrary functions.

From the Einstein equations for the background metric (3) follow the relations

$$R = \frac{10}{3}\Lambda - \frac{2}{3}\kappa^2\delta(y)(\tau_\mu^\mu - 4\hat{\sigma}) \simeq 20\mu^2 + 16\mu\delta(y) \quad (36)$$

$$R_\mu^\nu = \frac{1}{2}\delta_\mu^\nu(R - 2\Lambda) + \kappa^2\delta(y)(\tau_\mu^\nu - \hat{\sigma}\delta_\mu^\nu) \simeq \delta_\mu^\nu(-4\mu^2 + 2\mu\delta(y)) + \kappa^2\delta(y)\tau_\mu^\nu, \quad (37)$$

where in the radiation-dominated period we have set $\tau_\mu^\mu = 0$. The Ricci tensor calculated with the metric (33) is

$$R_\mu^\nu = -\frac{\delta_\mu^\nu}{2} \left(\frac{b''}{b} + \frac{b'^2}{b^2} \right) + \bar{R}_\mu^\nu(\bar{g}) = \delta_\mu^\nu(-4\mu^2 + 2\mu\delta(y)) + R_\mu^\nu(\bar{g}), \quad (38)$$

where we used the relations $b'/b = -2\mu \text{sign}(y)$ and $b''/b = 4\mu^2 - 4\mu\delta(y)$. Prime is derivative over y .

Using (27), we obtain the estimate of the components of the tensor $\bar{R}_\mu^\nu(\bar{g})$

$$R_\mu^\nu(\bar{g}) \sim 1/t^2 \sim \mu^2 \frac{\rho}{\mu}.$$

The components of the tensor $\kappa^2\tau_\mu^\nu$ are of order ρ . Comparing the expressions for R_μ^ν calculated with the exact and approximate metrics, we find that in the period of late cosmology when $\rho/\mu < 1$ and $\mu^2 > R_\mu^\nu(\bar{g})$, $\mu > \kappa^2\tau_\mu^\nu$, both expressions for the Ricci tensor coincide up to the terms of order ρ/μ , which is the precision of our calculations in the period of late cosmology.

Written with the approximate metric, the equations of motion are considerably simpler than with the exact metric. The (44) and (4 μ) and contracted ($\mu\nu$) components of equations of motion which follow from the action (28) are (cf. [17])

$$D^\mu D^\nu h_{\mu\nu} - D^2 h - \frac{3b'}{2b} h' = 0, \quad (39)$$

$$(D_\mu h - D^\nu h_{\mu\nu})' = 0 \quad (40)$$

$$D^\mu D^\nu h_{\mu\nu} - \frac{5b'}{2b} h' - h'' = 0, \quad (41)$$

where $h = h_{\mu\nu}\bar{g}^{\mu\nu}$. From the Eqs. (39) and (41) it follows that

$$h'' + \frac{b'}{b} h' = 0,$$

or $h'(y, x) = C(x)b^{-1}(y)$, where $C(x)$ is an arbitrary function. Because of the reflection symmetry $x \leftrightarrow -x$, the metric components $g_{\mu\nu}(y, x)$ and $h_{\mu\nu}(y, x)$ are even functions of y , and thus $h'(y, x)$ is an odd function of y . Because $b(y)$ is even in y , it follows that $C(x) = 0$ and $h'(y, x) = 0$. We obtain that on the equations of motion $h(y, x)$ is independent of y , $h(y, x) = h(x)$.

After imposing the gauge conditions $h_{4\mu} = 0$ (35), there remain residual gauge transformations $\xi_\mu(y, x) = b(y)C_\mu(x)$. Under these transformations the trace $h(x)$ transforms as $\delta h(x) = \bar{g}^{\mu\nu}(D_\mu C_\nu(x) + D_\nu C_\mu(x))$, and by a suitable choice of $C_\mu(x)$ can be transformed to zero.

From the Eq. (40) with $h = 0$ it follows that $D^\mu h'_{\mu\nu}(y, x) = 0$, or $D^\mu h_{\mu\nu}(y, x) = k_\nu(x)$. Using the remaining residual gauge transformations with the functions $C_\mu(x)$ such that $D^\mu C_\mu = 0$, we can impose the condition $D^\mu h_{\mu\nu}(y, x) = 0$.

5 Eigenmodes and eigenvalues

In the background of the approximate metric (33), in the gauge $D^\mu h_{\mu\nu}(y, x) = 0$, $h'_\mu(y, x) = 0$, the $(\mu\nu)$ components of the field equations for fluctuations are

$$h''_{\mu\nu} - 4\mu^2 h_{\mu\nu} + b^{-1}(y)D_\rho D^\rho h_{\mu\nu} + \delta(y)4\mu h_{\mu\nu} = 0. \quad (42)$$

Expanding the functions $h^\mu_\nu(x, y)$

$$h^\mu_\nu(x, y) = \sum_m \phi^\mu_{(m)\nu}(x) h_m(y),$$

where

$$D_\rho D^\rho \phi^\mu_{(m)\nu} = m^2 \phi^\mu_{(m)\nu},$$

and the eigenfunctions $h_m^<(y)$ are determined from the equation

$$h_m^{<''}(y) - 4\mu^2 h_m^<(y) + e^{2\mu|y|} m^2 h_m^<(y) + \delta(y)4\mu h_m^<(y) = 0. \quad (43)$$

In the region $y > \bar{y}$ in (22) the term with the increasing exponent becomes dominant. The approximate metric is

$$ds^2 = dy^2 + e^{2\mu y} \left(-dt^2 + \frac{a^2(0, t)}{4} \left(\frac{\rho}{\mu} \right)^2 \eta_{ab} dx^a dx^b \right). \quad (44)$$

The equation for the eigenmodes is

$$h_m^{>''}(y) - 4\mu^2 h_m^{>}(y) + e^{-2\mu|y|} m^2 h_m^{>}(y) = 0 \quad (45)$$

Eqs. (43) and (45) are solved in appendix B. We show that the norm of the function $h_m^{>}$ is smaller than that $h_m^{<}$. Effectively, this allows to neglect the contribution from the region $y > \bar{y}$. Solving the equations we neglect t dependence of $\bar{y}(t)$ which in the final expressions can be effectively sent to infinity. In the radiation-dominated period of late cosmology $e^{2\mu|\bar{y}|} < 1$ and $(d\bar{y}/dt)/\mu\bar{y} \ll 1$.

We obtain the spectrum

$$m_n \simeq \mu e^{-\mu\bar{y}} \left(n\pi + \frac{\pi}{2} \right) \quad (46)$$

and the normalized eigenmode $h_m(0)$

$$h_m(0) \simeq (\mu e^{-\mu\bar{y}})^{1/2}. \quad (47)$$

For the following we need the sum $\sum_n h_{m_n}^2(0)$, where m_n is determined by (46). Because of a narrow spacing between the levels, we change summation to integration and obtain

$$\sum_n h_{m_n}^2(0) \simeq \int \frac{dm}{\mu\pi} e^{\mu\bar{y}} \mu e^{-\mu\bar{y}} = \int \frac{dm}{\pi}. \quad (48)$$

The integral (48) is independent of \bar{y} . The same measure of integration was obtained in [11], where the authors used the graviton modes of the RS2 model without matter, in which case the integration over y extends to infinity and the spectrum is continuous. Similarity of the results can be traced to the fact that we performed calculations in the period of late cosmology neglecting the terms of order $O(\rho/\mu)$ as compared to unity. This result can be anticipated also from the equation for the eigenmodes which does not contain matter terms. Time (temperature) dependence of the Universe enters only through the \bar{y} , and calculations of appendix B are effectively performed by taking the limit $\bar{y} \rightarrow \infty$.

6 Production of Kaluza-Klein gravitons

Let us calculate the rate of production of Kaluza-Klein gravitons in interactions of particles in hot matter on the brane in the radiation-dominated period.

Production of Kaluza-Klein gravitons in reactions of particles localized on the brane is calculated with the interaction Lagrangian

$$I = \kappa \int d^4x \sqrt{-\bar{g}} h_{\mu\nu}(0, x) T^{\mu\nu}(x),$$

where $T^{\mu\nu}$ is the energy-momentum of particles on the brane. The Boltzmann equation which determines evolution of the energy density $\hat{\rho}(t)$ of matter on the brane is [20, 10, 12]⁷

$$\frac{d\hat{\rho}}{dt} + 4H\hat{\rho} = - \sum_n \sum_i \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k_1}{(2\pi)^3 2E_1} \frac{d^3k_2}{(2\pi)^3 2E_2} f_1^i(E_1) f_2^i(E_2) |M_n^i|^2 (2\pi)^4 \delta^4(k_1 + k_2 - p). \quad (49)$$

Here f^i are the Bose/Fermi distributions of colliding particles and M_n^i is the amplitude of reaction $\psi^i + \bar{\psi}^i \rightarrow G$, where ψ and $\bar{\psi}$ are standard model particles on the brane (vector, spinor, scalar) and G is a state of mass m_n from the graviton Kaluza-Klein tower⁸. The squared amplitude of annihilation into graviton is [10]

$$|M_n^i|^2 = A_i \frac{\kappa^2}{8} h_n^2(0) s^4, \quad (50)$$

where $A_i = A_s, A_v, A_f = 2/3, 4, 1$ for scalars, vectors and fermions and $s^2 = (k_1 + k_2)^2$. The sum over the graviton states is transformed to the integral following (48). After integration over the angular variables of the momenta of interacting particles Boltzmann equation takes the form

$$\begin{aligned} & \frac{d\hat{\rho}}{dt} + 4H\hat{\rho} \\ &= - \frac{\kappa^2 \sum_i A_i}{8(2\pi)^4} \int \frac{dm e^{\mu\bar{y}}}{\mu} h_m^2(0) m^4 \int dk_1 dk_2 f(k_1) f(k_2) (k_1 + k_2) \theta \left(1 - \left| 1 - \frac{m^2}{2k_1 k_2} \right| \right). \end{aligned} \quad (51)$$

⁷In this section we use the unnormalized energy density.

⁸We have written the rhs of (49) without the factor 1/2 (cf. [12]), because in annihilation reactions ψ^i and $\bar{\psi}^i$ are different particles [20, 21].

Integrating over m , we obtain

$$\begin{aligned} \frac{d\hat{\rho}}{dt} + 4H\hat{\rho} &= -\frac{\kappa^2 \sum_i A_i}{8(2\pi)^4} \int_0^{(2k_1 k_2)^{1/2}} dm m^4 \int dk_1 dk_2 f(k_1) f(k_2) (k_1 + k_2) \\ &= -\frac{\kappa^2 A 315 \zeta(9/2) \zeta(7/2) T^8}{\pi^3 2^8}, \end{aligned} \quad (52)$$

where the sum extends over relativistic degrees of freedom

$$A = \sum_i A_i = \frac{2g_s}{3} + g_f(1 - 2^{-7/2})(1 - 2^{-9/2}) + 4g_v. \quad (53)$$

In the high-energy period, when all the standard model degrees of freedom are relativistic, $A = 166, 2$. Eq.(52) has the same form as Eq.(18) and the expression in the rhs of (52) is the component \tilde{T}_{04} of the bulk energy-momentum tensor [12] (see (69) in the next section). The physical content of both equations is that the energy loss on the brane is due to graviton production.

7 The energy-momentum tensor in the bulk

Corrections to the leading-order solution to the Einstein equation are expressed through the components of the energy-momentum tensor of the emitted gravitons in the bulk. In this section we determine the y -dependence of the components of the energy-momentum tensor in the bulk in the period of late cosmology.

After being produced in reactions on the brane, gravitons move freely following the null geodesics in the bulk. Coordinates of particles on the geodesics are parametrized by an affine parameter λ . In the region $0 < y < \bar{y}$ we integrate geodesic equations using the approximate metric (33)

$$ds^2 = dy^2 + e^{-2\mu y} \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu.$$

Geodesic equations are

$$\frac{d^2 x^a}{d\lambda^2} + 2\Gamma_{by}^a \frac{dx^b}{d\lambda} \frac{dy}{d\lambda} + 2\Gamma_{b0}^a \frac{dx^b}{d\lambda} \frac{dx^0}{d\lambda} = 0 \quad (54)$$

$$\frac{d^2 x^0}{d\lambda^2} + 2\Gamma_{0y}^0 \frac{dx^0}{d\lambda} \frac{dy}{d\lambda} + 2\Gamma_{ab}^0 \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = 0 \quad (55)$$

$$\frac{d^2 y}{d\lambda^2} + \Gamma_{00}^y \left(\frac{dx^0}{d\lambda} \right)^2 + \Gamma_{ab}^y \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = 0, \quad (56)$$

where connections are

$$\Gamma_{by}^a = -\delta_b^a \mu, \quad \Gamma_{b0}^a = \delta_b^a \frac{\dot{a}}{a}, \quad \Gamma_{0y}^0 = -\mu, \quad \Gamma_{ab}^0 = \eta_{ab} \dot{a}, \quad \Gamma_{00}^y = -\mu e^{-2\mu y}, \quad \Gamma_{ab}^y = a^2 \eta_{ab} \mu e^{-2\mu y}.$$

Integrating the geodesic equations, we obtain

$$\frac{dx^a}{d\lambda} = \frac{C^a e^{2\mu y}}{a^2(0, t)} \quad (57)$$

$$\frac{dt}{d\lambda} = e^{2\mu y} \left(C_0^2 + \frac{C_a C^a}{a^2(0, t)} \right)^{1/2} \quad (58)$$

$$\left(\frac{dy}{d\lambda} \right)^2 - C_0^2 e^{2\mu y} = C_4. \quad (59)$$

Using these relations, it is straightforward to verify that

$$\left(\frac{dy}{d\lambda}\right)^2 + e^{-2\mu y} \left[-\left(\frac{dt}{d\lambda}\right)^2 + a^2 \eta_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right] = \text{const} \quad (60)$$

is independent of λ .

The energy-momentum tensor of the bulk gravitons is taken in the form of free radiation of massless particles [12]

$$\check{T}^{ij}(x) = \int d^5 p \sqrt{-g} \delta(p_i p^i) \tilde{f}(x, p) p^i p^j, \quad (61)$$

where $\tilde{f}(x, p)$ is the phase space density of the distribution function of gravitons. The function $\tilde{f}(x, p)$ is determined by identifying the interaction term in the rhs of the Boltzmann equation (52) and $2\check{T}_{04}$ in the $\{04\}$ component of the Einstein equations (18) written in the radiation-dominated period, $p = \rho/3$, in terms of the unnormalized energy density $\hat{\rho}$ [12]

$$\frac{d\hat{\rho}}{dt} + 4H\hat{\rho} = 2\check{T}_{04}.$$

The components of momentum p^i of gravitons subject to condition $p_i p^i = 0$ are proportional to the tangent vectors to null geodesics

$$p^0 = N(y, t) e^{2\mu y} \left(C_0^2 + \frac{C_a C^a}{a^2(0, t)} \right)^{1/2}, \quad p^4 = N(y, t) C_0 e^{\mu y}, \quad p^a(y) = N(y, t) \frac{C^a e^{\mu y}}{a(0, t)}, \quad (62)$$

where $N(y, t)$ is a normalization factor determined below. Integrating the geodesic equations (57)-(59) with the initial conditions

$$y(\lambda = 0) = 0, \quad t(\lambda = 0) = t_0, \quad x^a(\lambda = 0) = x^a,$$

we obtain $x^i = x^i(\lambda, C^i, t_0, x^a)$. Integrating (59), we have

$$\int_0^y dy' (C_4 + C_0^2 e^{2\mu y'})^{-1/2} = \lambda.$$

From this relation is determined $\lambda = \lambda(y, C^i)$. Inverting this relation, we obtain $y = y(\lambda, C^i)$. Because of the initial conditions, $y(0, C^i, x^a) = 0$.

The phase space density of non-interacting massless particles $\tilde{f}(x^i, p^i)$ as a function of λ is constant along a null geodesic

$$\frac{d\tilde{f}(x^i(\lambda), p^i(\lambda))}{d\lambda} = p^k \frac{\partial \tilde{f}}{\partial x^k} - \frac{\partial \tilde{f}}{\partial p^k} \Gamma_{rs}^k p^r p^s = 0. \quad (63)$$

Because the function $\tilde{f}(x^i(\lambda, C^i, t_0, x^a), p^i(\lambda, C^i, t_0, x^a)) \equiv \hat{f}(\lambda, C^i, t_0, x^a)$ is constant along geodesics, we have

$$\tilde{f}(x^i(\lambda, C^i, t_0, x^a), p^i(\lambda, C^i, t_0, x^a)) = \tilde{f}(0, t_0, x^a, p^i(0, C^i, t_0, x^a)) \equiv f(C^i, t_0, x^a).$$

Let us consider the $\{04\}$ component of the energy-momentum tensor

$$\check{T}^{04}(y, t, x^a) = \int d^5 p \sqrt{-g} \delta(p_i p^i) f(x, p) p^0 p^4.$$

Changing integration variables p^i to C^i , we obtain

$$\begin{aligned} \check{T}^{04}(y, t) = \int d^5 C^i e^{-4\mu y} a^3(0, t) \left| \frac{\partial(p^0, p^4, p^a)}{\partial(C^0, C^4, C^a)} \right| \delta(N^2 C_4) f(C^i, t_0, x^a) \\ N^2(y, t) e^{2\mu y} \left(C_0^2 + \frac{C_a C^a}{a^2(0, t)} \right)^{1/2} (C_0^2 e^{2\mu y} + C_4)^{1/2}. \end{aligned} \quad (64)$$

The Jacobian of the transformation is

$$\left(\frac{N e^{2\mu y}}{a(0, t)} \right)^3 \frac{N e^{2\mu y} C_0}{(C_0^2 + C_a C^a / a^2(0, t))^{1/2}} \frac{N}{2 (C_0^2 e^{2\mu y} + C_4)^{1/2}}. \quad (65)$$

We traced the distribution function along the geodesics to $\lambda = 0$, i.e. to the hypersurface $y = 0$. Separating the y -dependence, we have

$$\check{T}^{04}(y, t) = \check{T}^{04}(0, t) \left(\frac{N(y, t)}{N(0, t)} \right)^5 e^{6\mu y}. \quad (66)$$

Following [12], we expand the graviton momentum p^i in the orthonormal basis adapted to the foliation of the space-time

$$u^i = (1, 0, 0, 0, 0) e^{\mu y}, \quad n^i = (0, 0, 0, 0, 1), \quad e_a^i = (0, 1_a, 0, 0, 0) \frac{e^{\mu y}}{a(0, t)}. \quad (67)$$

We have

$$p^i = E u^i + m n^i + q^a e_a^i,$$

and $p_i p^i = -E^2 + m^2 + q_a q^a$.

Projections of the components of the bulk energy-momentum tensor on the plane $y = 0$

$$\check{T}_{\bar{0}\bar{4}} \equiv \check{T}^{ij} u_i n_j, \quad \check{T}_{\bar{0}\bar{0}} \equiv \check{T}^{ij} u_i u_j, \quad \check{T}_{\bar{4}\bar{4}} \equiv \check{T}^{ij} n_i n_j \quad (68)$$

have the form [12]⁹

$$\begin{aligned} \check{T}_{\bar{0}\bar{4}}(0, t) &= -\frac{315 A \zeta(9/2) \zeta(7/2)}{2^9 \pi^3} \kappa^2 T^8, \quad \check{T}_{\bar{4}\bar{4}}(0, t) = \frac{3 A \zeta(9/2) \zeta(7/2)}{2 \pi^4} \kappa^2 T^8, \\ \check{T}_{\bar{0}\bar{0}}(0, t) &= \frac{21 A \zeta(9/2) \zeta(7/2)}{8 \pi^4} \kappa^2 T^8, \end{aligned} \quad (69)$$

where T is the temperature of the Universe at the time t .

Separating in $\check{T}_{\bar{0}\bar{4}}$ the y -dependence, we have

$$\check{T}_{\bar{0}\bar{4}}(y, t) = \left(\frac{N(y, t)}{N(0, t)} \right)^5 e^{5\mu y} \check{T}_{\bar{0}\bar{4}}(0, t). \quad (70)$$

The same relations hold for all components: $\check{T}_{\bar{i}\bar{j}}(y, t) = (N(y, t) e^{\mu y} / N(0, t))^5 \check{T}_{\bar{i}\bar{j}}(0, t)$.

Our next aim is to determine the normalization factor $N(y, t)$. Lacking an independent definition of momentum of massless particle which could enable us to fix the normalization factor, let us consider

⁹Cf. (52).

the energy-momentum tensor conservation equations in the bulk $\nabla_i \check{T}^{ij}(y, t) = 0$ and try to extract from these equations an information on y -dependence of the energy-momentum tensor.

The component $j = 0$ of the conservation equations is

$$\nabla_i \check{T}^{i0}(y, t) = \partial_4 \check{T}^{40} + (3\Gamma_{04}^0 + \Gamma_{a4}^a) \check{T}^{40} + \partial_0 \check{T}^{00} + (2\Gamma_{00}^0 + \Gamma_{a0}^a) \check{T}^{00} + \Gamma_{ab}^0 \check{T}^{ab} = 0. \quad (71)$$

Substituting the explicit expressions for the Christoffel symbols, we obtain

$$\nabla_i \check{T}^{i0}(y, t) = \partial_4 \check{T}^{40} + \left(3\frac{n'}{n} + 3\frac{a'}{a}\right) \check{T}^{40} + \partial_0 \check{T}^{00} + \left(2\frac{\dot{n}}{n} + 3\frac{\dot{a}}{a}\right) \check{T}^{00} + \frac{\dot{a}a}{n^2} \eta_{ab} \check{T}^{ab} = 0, \quad (72)$$

where

$$\check{T}^{04} e^{-\mu y} = \check{T}_{\bar{0}\bar{4}}, \quad \check{T}^{00} e^{-2\mu y} = \check{T}_{\bar{0}\bar{0}}, \quad \check{T}^{44} = \check{T}_{\bar{4}\bar{4}}, \quad \check{T}^{ab} e^{-2\mu y} a^{-2}(0, t) = \check{T}_{\bar{a}\bar{b}}. \quad (73)$$

In the region $0 < y < \bar{y}$, using the approximate metric (25), we have

$$n'/n \simeq -\mu, \quad a'/a \simeq -\mu, \quad \dot{a}/a \simeq H(t), \quad \dot{n}/n \sim H(t)\rho/\mu.$$

Taking into account that in the radiation-dominated period $T \sim a(0, t)^{-1}$, we obtain that $\partial_0 \check{T}_{\bar{0}\bar{0}}(0, t) \simeq -8H(t) \check{T}_{\bar{0}\bar{0}}(0, t)$.¹⁰ Thus, Eq.(72) can be written as

$$(\partial_y \psi - 6\mu\psi) \check{T}_{\bar{0}\bar{4}}(0, t) + H e^{\mu y} \psi (-5 \check{T}_{\bar{0}\bar{0}}(0, t) + \eta_{ab} \check{T}_{\bar{a}\bar{b}}(0, t)) = 0, \quad (74)$$

where $\psi \equiv e^{\mu y} (e^{\mu y} N(y, t)/N(0, t))^5$. The trace of the energy-momentum tensor of massless particles is zero, $\check{T}_i^i = 0$. Using (68), we obtain a relation

$$\check{T}_{\bar{a}\bar{b}} \eta_{ab} = \check{T}_{\bar{0}\bar{0}} - \check{T}_{\bar{4}\bar{4}}.$$

Eq.(74) can be rewritten as

$$\partial_y \psi - 6\mu\psi + \psi H e^{\mu y} B = 0, \quad (75)$$

where $B = (4\check{T}_{\bar{0}\bar{0}} + \check{T}_{\bar{4}\bar{4}})/(-\check{T}_{\bar{0}\bar{4}}) \simeq 6.44$. Solving (75), we obtain

$$\left(\frac{N(y, t)}{N(0, t)}\right)^5 = \exp \left\{ -\frac{BH(t)}{\mu} (e^{\mu y} - 1) \right\}. \quad (76)$$

Next, let us consider the $j = 4$ component of the conservation equation of the energy-momentum tensor

$$\begin{aligned} \nabla_i \check{T}^{i4}(y, t) &= \partial_4 \check{T}^{44} + (\Gamma_{04}^0 + \Gamma_{a4}^a) \check{T}^{44} + \partial_0 \check{T}^{04} + (\Gamma_{00}^0 + \Gamma_{a0}^a) \check{T}^{04} + \Gamma_{00}^4 \check{T}^{00} + \Gamma_{ab}^4 \check{T}^{ab} \\ &= \partial_4 \check{T}^{44} + \left(\frac{n'}{n} + 3\frac{a'}{a}\right) \check{T}^{44} + \partial_0 \check{T}^{04} + \left(\frac{\dot{n}}{n} + 3\frac{\dot{a}}{a}\right) \check{T}^{04} + n' n \check{T}^{00} - a' a \eta_{ab} \check{T}^{ab} = 0 \end{aligned} \quad (77)$$

Repeating the same steps as for the component $j = 0$, we obtain

$$\partial_y \varphi - 5\mu\varphi - \varphi H e^{\mu y} \frac{5\check{T}_{\bar{0}\bar{4}}(0, t)}{\check{T}_{\bar{4}\bar{4}}(0, t)} = 0, \quad (78)$$

where $\varphi \equiv (N(y, t) e^{\mu y}/N(0, t))^5$ and $B' = -5\check{T}_{\bar{0}\bar{4}}/\check{T}_{\bar{4}\bar{4}} \simeq 6.21$. We have used that at $y = 0$ there is a relation

$$n' n \check{T}^{00} - a' a \eta_{ab} \check{T}^{ab} = -\mu(\check{T}_{\bar{0}\bar{0}} - \eta_{ab} \check{T}_{\bar{a}\bar{b}}) = -\mu \check{T}_{\bar{4}\bar{4}}.$$

Solving (77), we have

$$\left(\frac{N(y, t)}{N(0, t)}\right)^5 = \exp \left\{ -\frac{B'H(t)}{\mu} (e^{\mu y} - 1) \right\} \quad (79)$$

It is remarkable that the constants B and B' which appear in different equations are close to each other. The small difference between the constants can be attributed to approximations to the exact equations.

¹⁰We neglected a weak T dependence of $g_*(T)$.

8 Einstein equations with the bulk energy-momentum tensor

Having the expressions for the bulk energy-momentum tensor \tilde{T}^{ij} , we are in position to find the range of t and y within which solutions of the Einstein equations with the bulk energy-momentum tensor included are sufficiently close to the leading-order solutions, i.e. can be considered as perturbations. In particular, in Sect.4 we determined the spectrum of Kaluza-Klein tower of gravitons in the leading approximation which is possible, if $\rho/\mu \ll 1$ and if corrections to the leading approximation are small.

Let us consider the (0) component of the Einstein equations in the bulk

$$\frac{a''(y, t)}{a(y, t)} + \frac{a'^2(y, t)}{a^2(y, t)} - \frac{\dot{a}^2(y, t)}{n^2(y, t)a^2(y, t)} = 2\mu^2 + \frac{2\kappa^2}{3}\tilde{T}_0^0(y, t) \quad (80)$$

The ratio of the bulk term to $2\mu^2$ is

$$\frac{\kappa^2\tilde{T}_0^0(y, t)}{3\mu^2} = -\frac{\kappa^2\tilde{T}_{00}(y, t)}{3\mu^2} = \left(\frac{N(y, t)}{N(0, t)}\right)^5 (y)e^{5\mu y} \frac{\kappa^2\tilde{T}_{00}(0, t)}{3\mu^2} \quad (81)$$

where

$$\frac{\kappa^2\tilde{T}_{00}(0, t)}{3\mu^2} = \frac{21A\zeta(9/2)\zeta(7/2)}{24\pi^4} \left(\frac{180}{g_*\pi^2}\right)^2 \left(\frac{\rho}{\mu}\right)^2 \equiv K \left(\frac{\rho}{\mu}\right)^2. \quad (82)$$

Here we have used (19) to write

$$\kappa^2 T^4 = \frac{180\rho}{g_*(T)\pi^2},$$

where $g_* = g_s + g_v + 7g_f/8$. The coefficient K varies in the range from $K \simeq 0.2$ at characteristic energies of the nucleosynthesis, $T \sim 10^{-(3\div 4)}$ GeV, to $K = 0.025$ at the energies $\sim 10^{2\div 3}$ GeV at which all the standard model degrees of freedom are relativistic. In this range the ratio ρ/μ changes from $10^{-22\div 29}$ to $10^{(-3\div 1)}$. Expression (81) can be written as

$$\frac{\kappa^2\tilde{T}_0^0(y, t)}{3\mu^2} = K \left(\frac{\rho}{\mu}\right)^2 e^{5\mu y} \exp\left\{-\frac{HB}{\mu}(e^{\mu y} - 1)\right\}, \quad (83)$$

where $HB/\mu \simeq 9.1(\rho/\mu)^{1/2}$. In the region $y < \bar{y}$, where $e^{2\mu\bar{y}} = 1 + 2\mu/\rho$, for $\mu/\rho \ll 1$, we have

$$\frac{\kappa^2\tilde{T}_0^0(y, t)}{3\mu^2} < CK \left(\frac{\rho}{\mu}\right)^2,$$

where $C = O(1)$. Fixing t_l so that $\rho(t_l)/\mu \ll 1$, the ratio (81) can be made arbitrary small.

Integrating Eq.(7), we obtain the (04) component of the Einstein equations as

$$\frac{\dot{a}(y, t)}{n(y, t)} = \dot{a}(0, t) \exp\left(-\int_0^y \frac{\kappa^2\tilde{T}_{04}a}{3\dot{a}} dy'\right). \quad (84)$$

Using the formulas of Sect.7, we have

$$\begin{aligned} \int_0^y \frac{\kappa^2\tilde{T}_{04}(y', t)a}{3\dot{a}} dy' &= \int_0^y \frac{\kappa^2\tilde{T}_{04}(y', t)e^{-\mu y'}}{3H} dy' \\ &= \frac{\kappa^2\tilde{T}_{04}(0, t)}{3H} \int_0^y e^{4\mu y'} \exp\left\{-\frac{HB}{\mu}(e^{\mu y'} - 1)\right\} dy'. \end{aligned} \quad (85)$$

In the period of late cosmology, using the approximate metric (25), we can set $\dot{a}(y', t)/a(y', t) \simeq H(t)$. The factor at the integral (85) is

$$\frac{\kappa^2 \tilde{T}_{04}(0, t)}{3H} = \frac{315 A}{3\pi^3 2^9 \sqrt{2}} \left(\frac{180}{g_* \pi^2} \right)^2 \left(\frac{\rho(t)}{\mu} \right)^{3/2} \equiv K \left(\frac{\rho}{\mu} \right)^{3/2}. \quad (86)$$

At the temperatures of the Universe of order $10^{-3} GeV$, we have $A = 18$, $g_* = 10.75$ and $K \simeq 0.12$. At the temperatures at which all the Standard model degrees of freedom are relativistic, $K \simeq 0.011$. The integral in (85) is

$$I = \int_1^{e^{\mu y}} dx x^3 e^{-b(x-1)} = e^b \frac{6}{b^4} (f(b) - f(be^{\mu y})),$$

where $b = HB/\mu$ and

$$f(b) = e^{-b} (1 + b + b^2/2! + b^3/3!).$$

For $b < 1$ approximately $f(b) \simeq 1 - b^4/12$, and for $y < \bar{y}$ the integral is estimated as

$$I \simeq \frac{e^b}{2} (e^{4\mu y} - 1) < 2 \frac{\rho}{\mu} \exp \left\{ 9.1 \left(\frac{\rho}{\mu} \right)^{1/2} \right\}.$$

For small enough $\rho(t_l)/\mu$ the expression in the exponent in (84)

$$K \left(\frac{\rho}{\mu} \right)^{5/2} \exp \left\{ 9.1 \left(\frac{\rho}{\mu} \right)^{1/2} \right\} \quad (87)$$

can be made arbitrary small.

To conclude, contribution to the energy-momentum tensor due to graviton emission modifies solutions of the Einstein equations, however, in the period of late cosmology corrections to the leading-order solutions are numerically small as compared to the leading-order terms.

Let us consider the Friedmann equation (16) with the terms \tilde{T}_{ij} included. Calculating the first integral in (16), we obtain

$$I_{04} = \frac{2\kappa^2}{3a^4(0, t)} \int_{t_l}^t dt' \tilde{T}_{04}(0, t') (\mu + \rho(t')) a^{-4}(0, t') = \mu \rho(t) A_{04} \left(\frac{1}{12} \left(\frac{1}{\mu t_l} - \frac{1}{\mu t} \right) + \frac{1}{288} \left(\frac{1}{(\mu t_l)^3} - \frac{1}{(\mu t)^3} \right) \right), \quad (88)$$

where we have substituted (69)

$$\kappa^2 \tilde{T}_{04} = A_{04} \rho^2 = - \frac{315 A \zeta(9/2) \zeta(7/2)}{2^9 \pi^3} \left(\frac{180}{g_* \pi^2} \right)^2 \rho^2. \quad (89)$$

The integral (88) has a strong dependence on the value of the lower limit. Therefore, in the integrand the slowly varying functions can be taken at the times when all the Standard model degrees of freedom are relativistic. In this period $A = 166.2$, $g_*(T) = 106.7$ and $A_{04} \simeq -0.126$. In the leading approximation $\rho(t) \simeq 1/8\mu t^2$ and $1/\mu t_l \simeq (8\rho(t_l)/\mu)^{1/2}$. Taking $\rho(t_l)/\mu \sim 0.1 \div 0.001$ and $\mu \sim 10^{-12} GeV$, we have $1/\mu t_l \simeq 0.9 \div 0.09$ and $T_l \sim (5.1 \div 1.6) \cdot 10^2 GeV$. For $\rho(t_l)/\mu \sim 0.1$ we obtain

$$I_{04} \simeq -2\mu\rho \cdot 0.0048. \quad (90)$$

In the same way, calculating I_{44} , we obtain

$$I_{44} = \frac{2\kappa^2}{3a^4(0, t)} \int_{t_l}^t dt' \check{T}_{44}(t') (2\mu\rho(t'))^{1/2} a^{-4}(0, t') = \frac{\mu\rho(t)A_{44}}{48} \left(\frac{1}{(\mu t_l)^2} - \frac{1}{(\mu t)^2} \right), \quad (91)$$

where

$$\kappa^2 T_{44} = A_{44} \rho^2 = \frac{3 A \zeta(9/2) \zeta(7/2)}{4\pi^4} \left(\frac{180}{g_* \pi^2} \right)^2 \rho^2 \simeq 0.10 \rho^2.$$

This yields

$$I_{44} \simeq 2\mu\rho \cdot 0.00094. \quad (92)$$

9 Graviton emission in the bulk and nucleosynthesis

9.1 Graviton emission in the period of late cosmology

Let us compare abundances of ${}^4\text{He}$ produced in primordial nucleosynthesis calculated in the models with and without account of the graviton emission in the bulk. We consider the radiation-dominated period of late cosmology, in which production of gravitons is sufficiently intensive. In the period of late cosmology the leading approximation of the Friedmann equation coincides with the standard cosmological model.

We estimate the effect of the graviton emission on nucleosynthesis by solving Friedmann equation perturbatively keeping in the generalized Friedmann equation (16) the term linear in the radiation energy density and the terms containing the bulk energy-momentum tensor. The Weyl radiation term and the term quadratic in radiation energy density can be treated perturbatively also. We have

$$H^2 \simeq 2\mu\rho - I_{04} - I_{44}, \quad (93)$$

where the integrals I_{04} and I_{44} were defined in the previous section. We use also the 5D energy conservation equation (18) which is written as

$$\dot{\rho} + 4H\rho = -\frac{A_{04}\rho^2}{3}. \quad (94)$$

Let $\bar{\rho}$ and \bar{H} be the energy density and the Hubble function calculated in the model without inclusion in the Einstein equations the terms with the bulk energy-momentum tensor and the Weyl radiation. Defining

$$\rho = \bar{\rho} + \rho_1, \quad H = \bar{H} + H_1,$$

where $\bar{H}^2 = 2\mu\bar{\rho}$ and ρ_1 and H_1 are perturbations, and separating in (93) and (94) the leading-order terms, we obtain

$$2\bar{H}H_1 = 2\mu\rho_1 - \frac{2\bar{\rho}}{3} \int dt' [A_{04}(\mu + \bar{\rho}) + A_{44}\bar{H}] \bar{\rho}(t') \quad (95)$$

$$\dot{\rho}_1 + 4\bar{H}\rho_1 + 4\bar{\rho}H_1 = \frac{A_{04}\bar{\rho}^2}{3}. \quad (96)$$

Substituting in (95) $\bar{\rho}(t) \simeq 1/8\mu t^2$ and performing integration, we have

$$H_1(t) \simeq \frac{\mu}{\bar{H}}\rho_1 - \frac{\bar{H}A_{04}}{48\mu} \left[\frac{1}{t_l} - \frac{1}{t} + \frac{1}{24\mu^2} \left(\frac{1}{t_l^3} - \frac{1}{t^3} \right) \right] - \frac{\bar{H}A_{44}}{192\mu^2} \left(\frac{1}{t_l^2} - \frac{1}{t^2} \right). \quad (97)$$

Substituting expression (97) for H_1 in (96) and noting that $\bar{H} = 1/2t$, we obtain

$$\dot{\rho}_1 + \frac{3}{t}\rho_1 = \frac{A_{04}}{192\mu^2 t^3} \left[\frac{1}{t_l} + \frac{1}{24\mu^2} \left(\frac{1}{t_l^3} - \frac{1}{t^3} \right) \right] + \frac{A_{44}}{768\mu^3 t^3} \left(\frac{1}{t_l^2} - \frac{1}{t^2} \right).$$

Solving this equation, we find

$$\rho_1(t) = \frac{C_1}{t^3} + \frac{A_{04}}{192\mu^2 t^3} \left[\frac{t-t_l}{t_l} + \frac{1}{24\mu^2} \left(\frac{t-t_l}{t_l^3} - \frac{1}{2} \left(\frac{1}{t_l^2} - \frac{1}{t^2} \right) \right) \right] + \frac{A_{44}}{768\mu^3 t^3} \left(\frac{t}{t_l^2} - \frac{2}{t_l} + \frac{1}{t} \right) \quad (98)$$

The constant C_1 should be determined by sewing solutions of Friedmann equation in the periods of early and late cosmologies. For a moment we set $C_1 = 0$, i.e. look for a contribution from the period of late cosmology. For H_1 we obtain

$$H_1(t) = -\frac{A_{04}}{1536\mu^3 t^2} \left(\frac{1}{t_l^2} - \frac{1}{t^2} \right) + \frac{A_{44}}{192\mu^2} \left(-\frac{1}{t_l t^2} + \frac{1}{t^3} \right) \simeq -\bar{\rho}(t) \left(\frac{A_{04}}{192\mu^2 t_l^2} + \frac{A_{44}}{24\mu t_l} \right). \quad (99)$$

Expressions (98) and (99) show that corrections to the leading terms are small, i.e. the perturbative approach is justified. Note that the leading term proportional to $A_{04}/\mu t_l$ was canceled in H_1 .

The mass fraction of ${}^4\text{He}$ produced in primordial nucleosynthesis is [20, 21]

$$X_4 = \frac{2(n/p)}{(n/p) + 1}$$

where the ratio n/p is taken at the end of nucleosynthesis. Characteristic temperature of the onset of the period of nucleosynthesis (freezing temperature T_n of the reaction $n \leftrightarrow p$) is estimated as the temperature at which the reaction rate $\sim G_F T^5$ is approximately equal to the Hubble parameter [20]

$$G_F T_n^5 \sim H.$$

The difference of the freezing temperatures in the models with and without the account of the graviton emission is

$$\frac{\delta T_n}{T_n} \simeq \frac{H_1}{5\bar{H}} \simeq \frac{1}{5} \sqrt{\frac{\bar{\rho}(t_n)}{2\mu}} \left(\frac{A_{04}}{192\mu^2 t_l^2} + \frac{A_{44}}{24\mu t_l} \right), \quad (100)$$

where

$$\frac{\bar{\rho}(t_n)}{\mu} = \frac{8\pi^3 g_*(T_n)}{180} \frac{T_n^4}{\mu M^3}. \quad (101)$$

Substituting $T_n \sim 10^{-3} \text{GeV}$ and $\mu M^3 \sim (\mu M_{pl})^2 \sim 10^{14} \text{GeV}^4$, we find that the ratio $\delta T_n/T_n$ is very small. The equilibrium value of the $n-p$ ratio at the freezing temperature

$$\left(\frac{n}{p} \right)_n = \exp \left[-\frac{(m_n - m_p)}{T_n} \right]$$

is very sensitive to the value of T_n . Substituting $\delta(n/p)_n = (n/p) \ln(n/p)_n \delta T_n/T_n$, we obtain variation of X_4 under variation of the freezing temperature

$$\delta X_4 \simeq \frac{2}{(n/p + 1)^2} \ln \left(\frac{n}{p} \right) \left(\frac{n}{p} \right)_n \frac{\delta T_n}{T_n} \quad (102)$$

which is also a very small number.

9.2 Graviton emission in the period of early cosmology

Next, we make an estimate of the variation of X_4 due to the graviton emission in the bulk in the period of early cosmology, in which the ρ^2 term in the Friedmann equation is dominant. For the value of $\mu \sim 10^{-12} GeV$, the characteristic temperatures of this period are above $5 \cdot 10^2 GeV$.

To make an estimate of the graviton emission, we assume that the collision integral in the Boltzmann equation and the expressions for \check{T}_{ij} obtained for the period of late cosmology remain qualitatively valid in the early cosmological period. The new phenomenon in the early cosmological period is that some of the emitted gravitons can return to the brane and be again reflected in the bulk with a different momentum. These gravitons do not contribute to the component \check{T}_{04} , because they are not produced, but reflected, but contribute to the component \check{T}_{44} . The Friedmann equation contains a new term $\check{T}_{44}^{(b)}$ representing the energy-momentum tensor of gravitons bouncing back to the brane [10, 12]

$$H^2(t) = \rho^2(t) + 2\mu\rho(t) + \mu\rho_w(t) - \frac{2\kappa^2}{3a^4(0, t)} \int_{t_c}^t dt' \left[\check{T}_{04}(t')(\rho(t') + \mu) + \check{T}_{44}(t')H(t') - \check{T}_{44}^{(b)}(t')H(t') \right] a^4(0, t'). \quad (103)$$

Here the initial time t_c is the time of reheating. Numerical estimates [12] and considerations from the Vaidya model [11] suggest at $t \gg t_c$ the dominant contributions from the integrals of the components \check{T}_{ij} and the integral of the term $\check{T}_{44}^{(b)}$ are mutually canceled.

Let again $\bar{\rho}$ and \bar{H} be the energy density of matter on the brane and the Hubble function calculated in the model without the graviton emission in the period of early cosmology. We define

$$\rho = \bar{\rho} + \rho_2, \quad H = \bar{H} + H_2,$$

where in the period of early cosmology $\bar{\rho}(t) \simeq 1/4t$, and $\bar{H} \simeq \bar{\rho}(t)$ (see (27)). It is assumed that ρ_2 and H_2 are perturbations to the leading terms. From the Friedmann equation we have

$$2\bar{H}H_2 \simeq 2\bar{\rho}\rho_2 - \frac{2\bar{\rho}}{3} \int_{t_c}^t dt' \left[A_{04}(\mu + \bar{\rho}) + A_{44}\bar{H} - \kappa^2\check{T}_{44}^{(b)}\bar{H} \right] \bar{\rho}(t'). \quad (104)$$

The conservation law yields

$$\dot{\rho}_2 + 4\bar{H}\rho_2 + 4\bar{\rho}H_2 = \frac{A_{04}\bar{\rho}^2}{3}. \quad (105)$$

In the same way as in the period of late cosmology, we obtain the equation for ρ_2

$$\begin{aligned} \dot{\rho}_2 + \frac{2}{t}\rho_2 - \frac{1}{t} \left[\frac{A_{04}}{12} \left(\mu \ln \frac{t}{t_c} + \frac{1}{4} \left(\frac{1}{t_c} - \frac{1}{t} \right) \right) + \frac{A_{44}}{48} \left(\frac{1}{t_c} - \frac{1}{t} \right) \right] \\ + \frac{1}{12t} \int_{t_c}^t \kappa^2 \check{T}_{44}^{(b)}(t') dt' = \frac{A_{04}}{48t^2}, \end{aligned} \quad (106)$$

where we substituted $\bar{\rho} \simeq 1/4t$. Integrating (106) with the initial condition $\rho_2(t_c) = 0$, we obtain

$$\begin{aligned} \rho_2(t) = \frac{A_{04}\mu}{24} \left(\ln \frac{t}{t_c} - \frac{1}{2} + \frac{t_c^2}{2t^2} \right) + \frac{A_{04} + A_{44}}{96t_c} \left(1 - \frac{t_c^2}{t^2} \right) - \frac{A_{44}}{48} \left(\frac{1}{t} - \frac{t_c}{t^2} \right) \\ - \frac{1}{12t^2} \int_{t_c}^t dy y \int_{t_c}^y dx \kappa^2 \check{T}_{44}^{(b)}(x) \end{aligned} \quad (107)$$

The time t_c of reheating is estimated for the reheating temperature $T_R \sim 5 \cdot 10^6 \text{ GeV}$ [23]. From the relation $\bar{\rho}(t_c)/\mu \sim 1/4\mu t_c$ we have

$$\frac{1}{\mu t_c} \simeq \frac{16\pi^3 g_*(T_R) T_R^4}{90 M^3} \sim 4 \cdot 10^3,$$

where we have put $M \simeq (\mu M_{pl}^2)^{1/3} \sim 10^8 \text{ GeV}$ and $g_*(T_R) \sim 10^2$. It follows that $t_l/t_c \sim 1/\mu t_c \sim 10^{16}$.

In (107) there is a large term $(A_{04} + A_{44})/t_c$. Omitting the small terms, we have

$$\rho_2(t) \simeq \frac{A_{04}\mu}{24} \left(\ln \frac{t}{t_c} - \frac{1}{2} \right) + \frac{A_{04} + A_{44}}{96t_c} - \frac{A_{44}}{48} \frac{1}{t} - \frac{1}{12t^2} \int_{t_c}^t dy y \int_{t_c}^y dx \kappa^2 \check{T}_{44}^{(b)}(x) \quad (108)$$

On dimensional grounds the term $\kappa^2 \check{T}_{44}^{(b)}(t)$ at small t has the following structure

$$\kappa^2 \check{T}_{44}^{(b)}(t) = \frac{b_2}{t^2} + \frac{b_1\mu}{t} + b_0\mu^2 + \dots \quad (109)$$

Performing integration of the last term in (108) and taking $t \sim t_l$, we obtain

$$\rho_2(t_l) \simeq \frac{A_{04}\mu}{24} \left(\ln \frac{t_l}{t_c} - \frac{1}{2} \right) - \frac{A_{44}}{48t_l} + \frac{A_{04} + A_{44}}{96t_c} - \frac{b_2}{24t_c} + \frac{b_2}{12t_l} + \frac{b_1\mu}{24} \ln \frac{t_l}{t_c} \quad (110)$$

Next, we equate $\rho_2(t_l)$ and C_1/t_l^3 in (98). At the times $t \sim t_l$, where $\mu t_l \sim 1$, we have

$$C_1\mu^3 \simeq \mu \left(\frac{A_{04}}{24} \left(\ln \frac{1}{\mu t_c} - \frac{1}{2} \right) - \frac{A_{44}}{48} \right) + \frac{A_{04} + A_{44}}{96t_c} - \frac{b_2}{24t_c} + \frac{b_2\mu}{12} + \frac{b_1\mu}{24} \ln \frac{1}{\mu t_c}. \quad (111)$$

The term C_1/t^3 is

$$\frac{C_1}{t^3} \simeq \frac{1}{\mu^2 t^3} \left[\mu \left(\frac{A_{04}}{24} \left(\ln \frac{1}{\mu t_c} - \frac{1}{2} \right) - \frac{A_{44}}{48} \right) + \frac{A_{04} + A_{44}}{96t_c} - \frac{b_2}{24t_c} + \frac{b_2\mu}{12} + \frac{b_1\mu}{24} \ln \frac{1}{\mu t_c} \right]. \quad (112)$$

In the period of late cosmology the term C_1/t^3 generates in H_1 an additional contribution $\Delta H_1 \simeq \mu \Delta \rho_1 / \bar{H}$

$$\Delta H_1 = 16\bar{\rho} \left[\mu \left(\frac{A_{04}}{24} \left(\ln \frac{1}{\mu t_c} - \frac{1}{2} \right) - \frac{A_{44}}{48} \right) + \frac{A_{04} + A_{44}}{96t_c} - \frac{b_2}{24t_c} + \frac{b_2\mu}{12} + \frac{b_1\mu}{24} \ln \frac{1}{\mu t_c} \right], \quad (113)$$

where $\bar{\rho}(t) \simeq 1/8\mu t^2$ is the energy density in the period of late cosmology.

If the term $(A_{04} + A_{44})/96t_c \sim 3 \cdot 10^{-4}/t_c$ was not canceled, it would produce in ρ_1 the contribution

$$\Delta \rho_1 = \frac{A_{04} + A_{44}}{96t_c} \frac{1}{(\mu t_c)^3} \simeq \frac{3 \cdot 10^{-4}}{(\mu t)^3 t_c}.$$

From (97) we would have

$$\frac{\Delta H_1}{\bar{H}} = \frac{\mu}{\bar{H}^2} \Delta \rho_1 \simeq \frac{1.2 \cdot 10^{-3}}{(\mu t_c)(\mu t)}. \quad (114)$$

At time of the nucleosynthesis

$$\frac{1}{8(\mu t_n)^2} \simeq \frac{4\pi^3 g_*(T_n)}{90} \frac{T_n^4}{(\mu M_{pl})^2}.$$

For $T_n \sim 10^{-3} GeV$, we have $\mu t_n \sim 10^{12}$. From (114) we obtain $\Delta H_1(t_n)/\bar{H}(t_n) \sim 4$, which is too large a value, and would contradict the experimental data.

Assuming that the large terms in (113) are canceled, we have

$$\Delta H_1 = \frac{2\bar{\rho}(t_n)}{3} \left[A_{04} \left(\ln \frac{1}{\mu t_c} - \frac{1}{2} \right) + \frac{A_{04}}{2} + b_1 \ln \frac{1}{\mu t_c} \right]. \quad (115)$$

Because at the period of nucleosynthesis $\Delta H_1/\bar{H} \sim (\bar{\rho}(t_n)/\mu)^{1/2}$ is a small number, contribution from the early cosmology would result in a small variation of $\delta X_4/X_4$.

10 Conclusions and discussion

Calculations of this paper were performed under the restriction to the period of late cosmology, when $\rho(T)/\mu < 1$, where $\rho(T)$ is the normalized radiation energy density of matter on the brane, μ is the scale of the warping factor in the metric. For $\mu \sim 10^{-12} GeV$, which we adopted in this paper, the limiting temperatures of the Universe at which the approximation of late cosmology is valid are of order $T_l \sim 5 \cdot 10^2 GeV$.

In the period of late cosmology it was possible to make a number of approximations, which enabled us to obtain the analytic expression for the energy loss from the brane to the bulk due to graviton emission. We calculated the spectrum of gravitons and found the energy-momentum tensor of emitted gravitons in the bulk. The Einstein equations were solved perturbatively, taking as the leading-order solution that without the graviton emission. Using the expressions for the energy-momentum tensor of gravitons in the bulk, we estimated corrections to the leading-order solution to the Einstein equations and showed that the perturbative approach is justified. We estimated also corrections to the leading-order terms in the generalized Friedmann equation and showed that in the late cosmological period they are sufficiently small as compared to the leading-order terms.

Graviton emission changes the cosmological evolution of matter on the brane and thus production of light elements in primordial nucleosynthesis. Solving the system of the generalized Friedmann and the 5D energy conservation equations, which included the terms representing the energy flux from the brane to the bulk, we found the difference of the mass fractions of 4He produced in primordial nucleosynthesis calculated in the models with and without the graviton emission

$$\frac{\delta X_4}{X_4} \sim \alpha \sqrt{\frac{\bar{\rho}(T_n)}{\mu}}.$$

Here $\bar{\rho}$ is the radiation energy density on the brane in the leading approximation, without account of graviton emission, $T_n \sim 10^{-3} GeV$ is the freezing temperature of the reaction $p \leftrightarrow n$, and $\alpha \ll 1$ is a small number. The ratio $\rho(T_n)/\mu \sim 10^{-26}$ is a very small number, and thus the difference of abundances of 4He calculated in both models is small. Crude estimate of the contribution to $\delta X_4/X_4$ from the early cosmological period based on the assumption that in the Friedmann equation the large contributions from the terms containing the energy-momentum tensor of the emitted gravitons cancel due to the bounce of the emitted gravitons back to the brane [12] again shows that it is a small number.

In papers [10, 12] as a measure of energy density loss on the brane due to graviton production was introduced the integral

$$\Omega_{lost} = \int_{t_i}^t dt \frac{-\Delta \dot{\hat{\rho}}}{\hat{\rho}},$$

where $-\Delta\dot{\hat{\rho}}$ is the collision integral in the rhs of the Boltzmann Eq.(52) which gives the rate of graviton emission. Because in [10, 12] and in the present paper the collision integrals are identical (up to the factor 1/2), expressions for Ω_{lost} up to the factor 1/2 are the same. Substituting expressions for $\Delta\dot{\hat{\rho}}$ and $\dot{\hat{\rho}}$, we find

$$\Omega_{lost} = \frac{A_{04}}{24\mu} \left(\frac{1}{t_l} - \frac{1}{t} \right),$$

where t_l is the limiting time at which approximation of the late cosmology is valid, $\mu t_l \sim 1$, and $A_{04} = -0.126$. However, calculating correction $H_1(t)$ to the leading-order Hubble function $\bar{H}(t) = 1/2t$, we found that in $H_1(t)$ the terms $H(t)A_{04}/\mu t_l$ are canceled. The ratio of the would-be correction $H(t)A_{04}/48\mu t_l$ to the actual correction $\rho(t)(A_{04}/192\mu^2 t_l^2 + A_{44}/24\mu t_l)$ is of order $(\rho/\mu)^{1/2}$. As discussed above, at the temperatures of nucleosynthesis this is a small number.

An alternative approach to calculation of emission of the Kaluza-Klein gravitons from the brane to the bulk was developed in [13, 14] and refs. therein. In this scenario there are a visible brane moving in the AdS bulk and a static brane. The motion of the visible brane is determined by evolution of matter on the brane. The moving brane acts as a time-dependent boundary to the 5D bulk leading to production of gravitons via the Casimir effect. Although this picture in appearance is very different from that of the present work, many of the intermediate formulas and final expressions are rather similar. In particular, the energy density of the emitted Kaluza-Klein gravitons on the brane was calculated to be $\rho_{KK} \sim 1/a^6$, where $a(t)$ is the scale factor in the induced metric on the brane. The energy density loss on the brane is $d\rho_{KK}/dt \sim \dot{a}/a^7$. In the radiation-dominated period of late cosmology, using the Friedmann equation $(\dot{a}/a)^2 \simeq 2\mu\rho$, it is obtained that $T \sim const/a(t)$ [20, 21] and $d\rho_{KK}/dt \sim T^8$ in correspondence with the present work and [10, 11, 12].

Graviton emission in the bulk was estimated also in the framework of DGP-like models. In the model with the action [24]

$$S = M^3 \int d^4x \int_0^R dy \sqrt{-g^{(5)}} R^{(5)} + M^3 r_c \int d^4x \sqrt{-g^{(4)}} R^{(4)}$$

where M is in the TeV range, $M_{pl}^2 = M^3(R + r_c)$, the ratio of the rate of the graviton emission in the bulk to the cooling rate of matter due to cosmological expansion is

$$\left. \frac{d\rho}{dt} \right|_{em} / \left. \frac{d\rho}{dt} \right|_{exp} \sim \frac{T}{M_{pl}} \alpha,$$

where $\alpha \sim 10^{-4}$.

In [25] was proposed a model with the action

$$S = M_*^3 \int d^4x dy \sqrt{-g^{(5)}} R^{(5)} + M_{pl}^2 \int d^4x \sqrt{-g^{(4)}} R^{(4)} + \int d^4x \sqrt{-g^{(4)}} L_{SM},$$

where $M_* \sim 10^{-12} GeV$. The strong gravitational coupling, $1/M_*^2$, presumably, is renormalized by the loops of the standard model fields down to $1/M_{pl}^2$. The ratio of the rate of the graviton emission to the cooling rate of matter was estimated as

$$\left. \frac{d\rho}{dt} \right|_{em} / \left. \frac{d\rho}{dt} \right|_{exp} \sim \frac{T^3}{M_{pl} M_*^2}.$$

This ratio is less than unity for $T < 10^2 GeV$.

A 3D brane in 5D bulk

From the system of 5D Einstein equations and junction conditions follows the 4D form of the Einstein equations

$$G_{\mu\nu}^{(4)} = \kappa_4^2 T_{\mu\nu}^{(4)},$$

where the Einstein tensor $G_{\mu\nu}^{(4)}$ is formed with the 4D metric $g_{\mu\nu}$, and $\kappa_4^2 = \kappa^2 \mu$. Explicitly [5]

$$\kappa_4^2 T_{\mu\nu}^{(4)} = \mu \kappa^2 \tau_{\mu\nu} + \kappa^4 \pi_{\mu\nu} + \frac{2\kappa^2}{3} \left(\check{T}_{\mu\nu} + \left(\check{T}_{44} - \frac{1}{4} \check{T}_i^i \right) g_{\mu\nu} \right) - E_{\mu\nu}, \quad (116)$$

where we have set $\sigma = \mu$. Here $\pi_{\mu\nu}$ is quadratic in $\tau_{\mu\nu}$, and $E_{\mu\nu}$ is the 'electric' part of the 5D Weyl tensor [5, 7]. The component $T^{(4)0}_0$ is

$$\kappa_4^2 T^{(4)0}_0 = -6\mu\rho - 3\rho^2 - \mu\rho_D, \quad (117)$$

where

$$\mu\rho_D = -\frac{2\kappa^2}{3} \left(\check{T}_0^0 + \check{T}_{44} - \frac{\check{T}}{4} \right) - E_0^0.$$

Noting that $G^{(4)0}_0 = -3H^2$, we obtain the generalized Friedmann equation

$$H^2 = 2\mu\rho + \rho^2 + \frac{\mu\rho_D}{3}. \quad (118)$$

From the 4D Bianchi identity $D_\mu G^{(4)\mu}_\nu = 0$ follows the 4D conservation law $D_\mu T^{(4)\mu}_\nu = 0$. The zero component of the conservation law, $D_\mu T^{(4)\mu}_0 = 0$, is

$$\dot{T}^{(4)0}_0 + 4HT^{(4)0}_0 - HT^{(4)\mu}_\mu = 0. \quad (119)$$

Substituting in this equation $T^{(4)0}_0$ from (117) and expressing $\dot{\rho}$ via (18), we obtain (cf. [9, 12])

$$\dot{\rho}_D + 4H\rho_D + \frac{2\kappa^2}{\mu} [\check{T}_{04}(\mu + \rho) + H\check{T}_{44}] = 0. \quad (120)$$

B Eigenmodes and eigenvalues

Solution of the equation

$$h_m''(y) - 4\mu^2 h_m(y) + e^{2\mu|y|} m^2 h_m(y) = 0 \quad (121)$$

is

$$h_m(y) = C_1 J_2(\tilde{m} e^{\mu|y|}) + C_2 N_2(\tilde{m} e^{\mu|y|}), \quad (122)$$

where

$$\tilde{m} = \frac{m}{\mu}.$$

The term with $\delta(y)$ is taken into account by the boundary condition

$$\left[\frac{dh_m(y)}{dy} + \frac{2}{\tilde{m}} h_m(y) \right]_{y=0+} = 0$$

which yields the relation

$$C_1 J_1(\tilde{m}) + C_2 N_1(\tilde{m}) = 0.$$

Using this relation we obtain the eigenfunctions in the region $y < \bar{y}$ the form

$$h_{m<}^<(y) = C \left[(N_1(\tilde{m})J_2(\tilde{m}e^{\mu|y|}) - J_1(\tilde{m})N_2(\tilde{m}e^{\mu|y|})) \right]. \quad (123)$$

¹¹ Introducing the functions

$$f_k(y) = N_1(\tilde{m})J_k(\tilde{m}e^{\mu|y|}) - J_1(\tilde{m})N_k(\tilde{m}e^{\mu|y|})$$

and using the formula

$$\int dx x Z_n^2(ax) = \frac{x^2}{2} [Z_n^2(ax) - Z_{n+1}(ax)Z_{n-1}(ax)], \quad (124)$$

where Z is any linear combination of the Bessel functions, we obtain

$$||h_m^<||^2 = 2C^2 \int_0^{\bar{y}} e^{2\mu y} f_2^2(y) dy = \frac{C^2}{\mu} [e^{2\mu\bar{y}} (f_2^2(\bar{y}) - f_1(\bar{y})f_3(\bar{y})) - (f_2^2(0) - f_1(0)f_3(0))]. \quad (125)$$

Typical masses (energies) of the emitted Kaluza-Klein gravitons are of order of temperature of the Universe T . In the case of $\mu \sim 10^{-12}$ GeV we have $m/\mu \gg 1$.

For large values of arguments of the Bessel functions, we obtain

$$\begin{aligned} f_1(y) &\simeq \left(\frac{e^{-\mu y}}{\pi \tilde{m}} \right)^{1/2} [-N_1(\tilde{m}) \cos z + J_1(\tilde{m}) \sin z] \\ f_2(y) &\simeq \left(\frac{e^{-\mu y}}{\pi \tilde{m}} \right)^{1/2} [N_1(\tilde{m}) \sin z + J_1(\tilde{m}) \cos z] \\ f_3(y) &\simeq \left(\frac{e^{-\mu y}}{\pi \tilde{m}} \right)^{1/2} [N_1(\tilde{m}) \cos z - J_1(\tilde{m}) \sin z], \end{aligned} \quad (126)$$

where $z = \tilde{m}e^{\mu y} - \pi/4$. For large \tilde{m} , substituting asymptotics of the Bessel functions, we have $J_1^2(\tilde{m}) + N_1^2(\tilde{m}) \simeq 2/\pi\tilde{m}$. At the upper limit of integration we obtain

$$f_2^2(\bar{y}) - f_1(\bar{y})f_3(\bar{y}) \simeq \left(\frac{2}{\pi\tilde{m}} \right)^2 e^{-\mu\bar{y}}.$$

At the lower limit we have

$$f_2^2(0) = \left(\frac{2}{\pi\tilde{m}} \right)^2, \quad f_1(0)f_3(0) = 0.$$

Combining the above expressions, we have

$$||h_m^<||^2 = \frac{C^2}{\mu} \left(\frac{2}{\pi\tilde{m}} \right)^2 (e^{\mu\bar{y}} - 1). \quad (127)$$

In the region $y > \bar{y}$ in (22) the term with the increasing exponent becomes dominant. We obtain the equations for the eigenmodes

$$h_{m>}^{>''}(y) - 4\mu^2 h_m^<(y) + e^{-2\mu|y|} m_{>}^2 h_{m>}^>(y) = 0 \quad (128)$$

¹¹To simplify the formulas we frequently omit the sub(superscripts) $>$ and $<$.

with solutions

$$h_{m>}^>(y) = \tilde{C}_1 J_2(\tilde{m}e^{-\mu y}) - \tilde{C}_2 N_2(\tilde{m}e^{-\mu y}). \quad (129)$$

For large y , such that $\tilde{m}e^{-\mu y} \ll 1$, the function $N_2(\tilde{m}e^{-\mu y}) \sim (\tilde{m}e^{-\mu y})^{-2}$ rapidly increases, and to have normalizable eigenfunctions we set $\tilde{C}_2 = 0$. The norm of the function $h_{m>}^>(y)$ is

$$||h_{m>}^>||^2 = \tilde{C}_1^2 \int_{\bar{y}}^{\infty} e^{-2\mu y} J_2^2(\tilde{m}e^{-\mu y}) dy = \tilde{C}^2 \frac{e^{-2\mu \bar{y}}}{2\mu} [J_2^2(\tilde{m}e^{-\mu \bar{y}}) - J_1(\tilde{m}e^{-\mu \bar{y}}) J_3(\tilde{m}e^{-\mu \bar{y}})]. \quad (130)$$

At temperatures $T \gg \mu$ at which production of gravitons is sufficiently intensive $\tilde{m}e^{-\mu \bar{y}} \sim (T/\mu)(\rho/\mu)^{1/2} \gg 1$, and we can substitute the asymptotics of the Bessel functions¹².

Asymptotic of the eigenfunction (123) is

$$h_m^>(y) = \frac{2C}{\pi \tilde{m}} e^{-\mu \bar{y}/2} \cos(\tilde{m}(e^{\mu \bar{y}} - 1)). \quad (131)$$

Instead of sewing the oscillating functions $h_m(y)$ and $h_m^>(y)$, we sew the envelopes of their asymptotics

$$\frac{2C}{\pi \tilde{m}} e^{-\mu \bar{y}/2} = \tilde{C}_1 \left(\frac{2e^{\mu \bar{y}}}{\pi \tilde{m}} \right)^{1/2} \quad (132)$$

giving

$$\tilde{C}_1 = C \left(\frac{2}{\pi \tilde{m}} \right)^{1/2} e^{-\mu \bar{y}}$$

Using this relation, we obtain

$$||h_m^>||^2 \simeq \frac{\tilde{C}_1^2}{\mu} e^{-2\mu \bar{y}} \frac{2e^{\mu \bar{y}}}{\pi \tilde{m}} = \frac{C^2}{2\mu} \left(\frac{2}{\pi \tilde{m}} \right)^2 e^{-\mu \bar{y}}. \quad (133)$$

Because $e^{\mu \bar{y}} > 1$, the norm (133) is smaller than (127).

Effectively, we neglect the contribution from the region $y > \bar{y}$ and impose the condition $h_m(\bar{y}) = 0$. Substituting $h_m(\bar{y})$ from (131), we have

$$\cos(\tilde{m}(e^{\mu \bar{y}} - 1)) = 0.$$

From this equation we obtain the spectrum

$$m_n \simeq \mu e^{-\mu \bar{y}} \left(n\pi + \frac{\pi}{2} \right). \quad (134)$$

Normalization constant C determined from condition $||h_m|| = 1$ with the norm (127) is

$$C \simeq \frac{\pi \tilde{m}}{2} \mu^{1/2} e^{-\mu \bar{y}/2}. \quad (135)$$

The normalized eigenmode $h_m(0)$ is

$$h_m(0) = \frac{2C}{\pi \tilde{m}} \simeq (\mu e^{-\mu \bar{y}})^{1/2}. \quad (136)$$

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¹²Taking $T \sim 10^2 \text{ GeV}$, we obtain from (19) $\rho/\mu \sim 10^{-4}$ and $T/\mu \sim 10^{10}$.

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